Bezier curves: A classroom investigation

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Pierre Bezier was born on September 1 in Paris, 1910. Choosing the career path of his father, he gained electrical and mechanical engineering degrees by the time he was 21 years old. He began working for the car company Renault and stayed there for the next 42 years. In 1977 he received a doctorate in mathematics. Bezier, at the age of 50 began developing an interest in drawing machines — interactive free form curve and surface design. Today, his system forms the basis of much of the computer aided design packages available in the market place.

Like a lot of great ideas, the basis of much of Bezier's early work (Devroye, 2005) is not difficult to understand. A brief investigation led me quickly down the path of binomial expansions, parametric equations, the concept of nesting, and spinners/scroll bars in an Excel spreadsheet. The ideas presented here could easily be incorporated into the classroom as a creative activity for upper secondary students.

I can vaguely recall some years ago an article written by the Martin Gardner (1959) of *Scientific American* fame[1], where the path of four dogs, one at each corner of a square room, was described as they set off chasing each others tails. The dogs begin to move along the wall, toward the tail of the dog directly in front of them. However almost immediately the dogs realise that their target tail is moving. This causes the dogs to spiral logarithmically toward each other into the inevitable melee at the centre of the room. Perhaps this metaphor best describes the sense of a Bezier curve, if not its mathematics.

To understand Bezier's wonderful insight as it would apply to the dogs, we need to simplify the chase by assuming that there are only two dogs in the room, one in the South West corner (say corner A), and one in the North West corner (say corner B) (see Figure 1).

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1. Martin Gardner had an enormous influence on my choice of career as a mathematics teacher. As a high school student in the seventies, I was an avid reader of "Mathematical Games" in the monthly editions of *Scientific American.* His books on recreational mathematics are now commonplace in mathematics staff rooms around the world. The problem is also illustrated at www.mathsworld.wolfram.com/miceproblem.html.

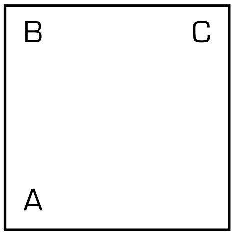

Figure 1. Diagram of the setup for two dogs.

The dog at corner B has some reason to move to the north-east corner C (perhaps he is hungry and sees a delicious bone). Let us also suppose that the dog at corner A takes off after the hungry dog. The path he would follow would resemble a quarter circle from A to C. We can determine an equation for this Bezier curve by looking at the two straight-line paths between these corners. If the dog in corner A were to run directly with uniform speed to B in unit time, the path would be given by $P_1(t) = A(1-t) + Bt$, where P_1, A and B are position vectors. At time $t = 0$, the dog would be at $P(0) = A$, and at time $t = 1$, the end of the journey, the dog would be at $P(1) = B$. The path from B to C would similarly be described by $P_1(t) = B(1-t) + Ct$.

The insight that Pierre Bezier collared was that the dog's path would be a nested relationship containing these two straight-line paths.

To describe the dog's path, Bezier would have formed the equation:

$$P_2(t) = \left[A(1-t) + Bt\right](1-t) + \left[B(1-t) + Ct\right]t \qquad (1)$$

Note carefully how each of the straight-line functions are nested as components in the equation. At $t = 0$, the equation tells us that the dog is indeed at A, and at $t = 1$ the dog is at C. At the halfway point, with

$$t = \frac{1}{2}, \quad P(t) = \frac{A + 2B + C}{4}$$

Using the simple coordinates A (0,0), B (0,1) and C (1,1) we can pinpoint this half way position as

$$x = \frac{0 + 2(0) + 1}{4} \text{ and } y = \frac{0 + 2(1) + 1}{4} \text{ or } \left(\frac{1}{4}, \frac{3}{4}\right)$$

Equation (1) can be simplified to:

$$P_2(t) = A(1-t)^2 + 2B(1-t)t + Ct^2 \qquad (2)$$

The result remarkably produces the binomial coefficients, and this if you like is a "phenotypic" response to the equation's symmetry.

Keeping with our (0,0), (0,1), (1,1) coordinates, we see that at anytime t, $0 \le t \le 1$:

$$x = t^2 \text{ and } y = 2(1-t)t + t^2 = 2t - t^2 \qquad (3)$$

These parametric equations show that the actual path of the dog is given by $y = 2\sqrt{x} - x$, which could be rearranged to $x^2 + y^2 = 2x(2 - y)$ showing us that

the path is not quite the quarter circle we envisaged (i.e., $x^2 + y^2 = 2x$). Figure 2 shows the graph of the dog's journey[2]:

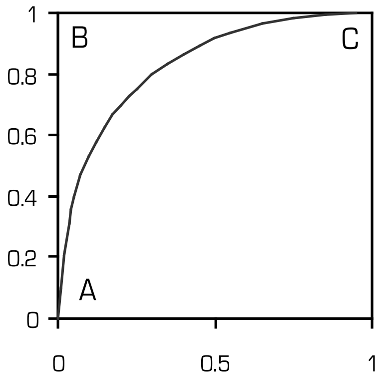

Figure 2. Graph of the dog's journey.

The path is in fact a parabolic section[2], with

$$y' = \frac{1}{\sqrt{x}} - 1$$

(see footnote, unless you derived this). The derivative displays Bezier's discovery about the tangents to a Bezier curve at its end points. In our simple example, the dog's initial movement is in the direction along the wall, and its final movement is in the direction of the other wall. The beauty of this is that two Bezier curves can be joined together smoothly.

If we widen our exploration by considering general coordinates A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3), we can graph the dog's position across the unit time interval on an Excel spreadsheet with:

$$P_{2,x}(t) = x_1(1-t)^2 + 2x_2(1-t) + x_3t^2 \text{ and } P_{2,y}(t) = y_1(1-t)^2 + 2y_2(1-t) + y_3t^2 \quad (4)$$

The corners A and C are actually known as *terminals*, and B is known as a *control point*. If we now place Excel "spinners" on the three coordinates, we can witness what Pierre Bezier witnessed in the early 1960s when he was searching for a way to control the shape of computer generated curves. Bezier had been conducting design research with the Renault automobile company for which he worked. If we come "out of the corners" and into the real ring, we might choose terminals and control points anywhere on the Cartesian plane.

Figures 3a and 3b show Bezier curves with two different control points and terminals (3,4) and (18,4):

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2. The dog's path is a section of the parabola $(x + y)^2 = 4x$. Under a 45° anticlockwise axes rotation, using $y = \frac{1}{\sqrt{2}}(x' - y')$ and $y = \frac{1}{\sqrt{2}}(x' + y')$, the parabola takes the more familiar form $y = x - \frac{1}{\sqrt{2}}(x^2)$. Furthermore, $(x + y)^2 = 4x$ has the single solution (0,0) when $y = -x$. This means that the line $y = -x$ must be parallel to the axis of symmetry. It is also a remarkable fact that the dog's path is that section of the parabola lying above the latus rectum.

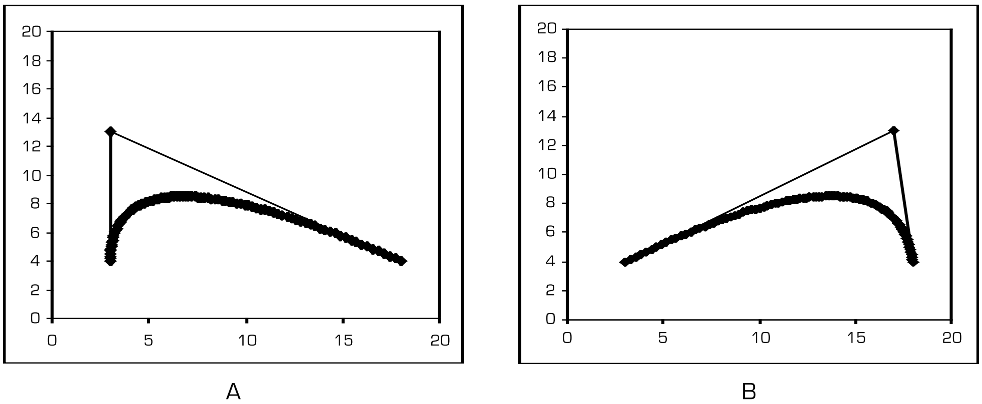

Figure 3. Bezier curves with terminals (3,4) and (18,4) and control points at (a) (3, 13) and (b) (17, 13).

If we look carefully at expression (2), we see that the coefficients of A, B and C are the *continuous* equivalent of the discrete binomial probabilities familiar to students. The coefficients are generally known as *Bernstein Polynomials* of order n having the form:

$$B(n,r)={}^{n}C_{r}\,t^{r}\,(1-t)^{(n-r)} \qquad 0 \le t \le 1 \qquad n \in J^{+} \quad t \in \Re \tag{3}$$

The order n signifies that there are n points in the Bezier construction, two terminals and $(n-2)$ control points. There are of course $(r+1)$ terms in the general Bezier expression and, for example, $B(3,1) = 3t(1-t)^2$ becomes the coefficient of the first control point for $P_3(t)$.

In a sense, the quantity $B(n,r)$ gives us an idea of the weighting being placed on the nth control point which is varying across the interval $0 < t < 1$. The graph of $B(3,1)$ in Figure 4 shows the change in weighting of this coefficient across unit time. In this case, at $t = 0.3$, the weighting of the first control point on $P_3(t)$ reaches a maximum of 0.44. Note, the Bernstein polynomials can be investigated with spinners and scroll bars as well (see www.canberramaths.org.au).

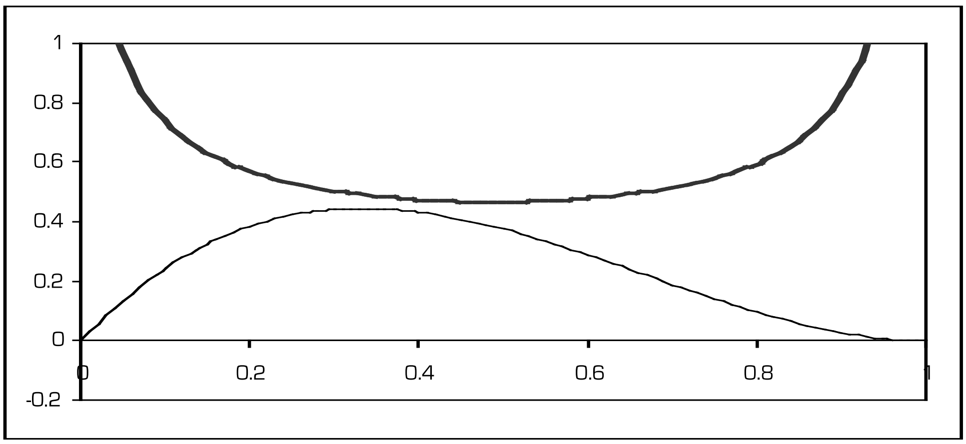

Figure 4. The graph for $B(3,1)$.

The "U" shaped curve above the Bernstein graph in Figure 4 is the envelope of all Bernstein polynomials of order n, and is given by:

$$f(t) = \frac{1}{\sqrt{2\pi n t(1-t)}}$$

If we extend the Bezier from one control point to two control points we increase our power to control the Bezier curve's shape. If you like, it enables another degree of freedom. By employing the Bernstein polynomials, we find that for two control points:

$$P_3(t) = A(1-t)^3 + 3B(1-t)^2 t + 3C(1-t)t^2 + Dt^3 \tag{4}$$

Again the binomial coefficients are present. Equation (4) reveals itself because of the simple principle of nesting as illustrated by:

$$P_3(t) = \left[A(1-t)^2 + 2B(1-t)t + Ct^2\right](1-t) + \left[B(1-t)^2 + 2C(1-t)t + Dt^2\right]t$$

Figure 5 is an example of a four point Bezier, with two terminals and two control points.

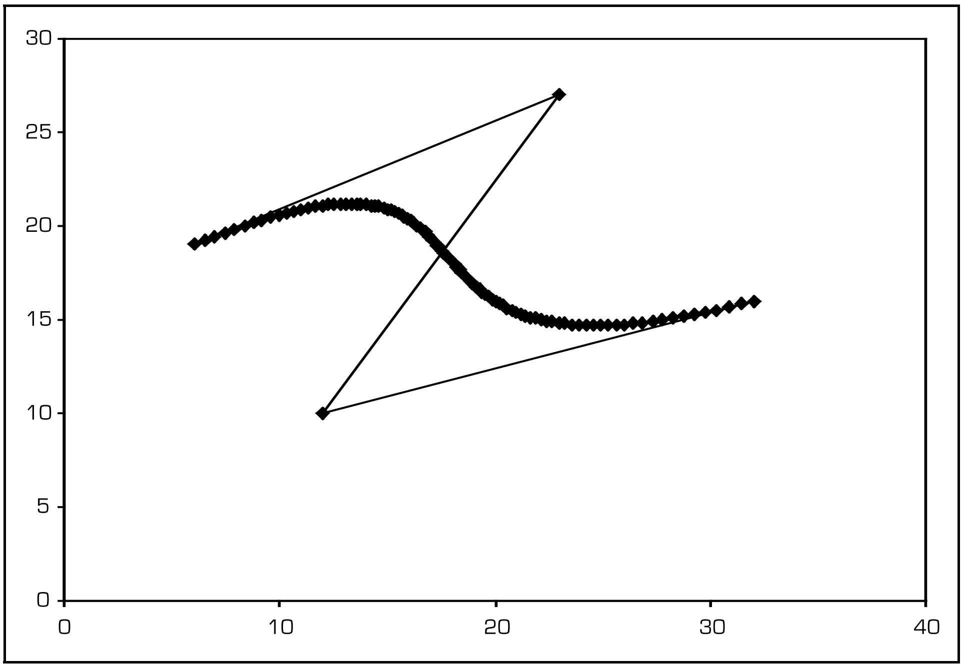

Figure 5. A Bezier curve with two terminals and two control points.

The number of control points is limitless, and the generated Bezier can close, loop back on itself, or pulled into any shape that the controller desires. Figure 6 shows a five point closed Bezier that also contains a loop. Try to imagine the path of the dog from the point (3, 11) to the same point!

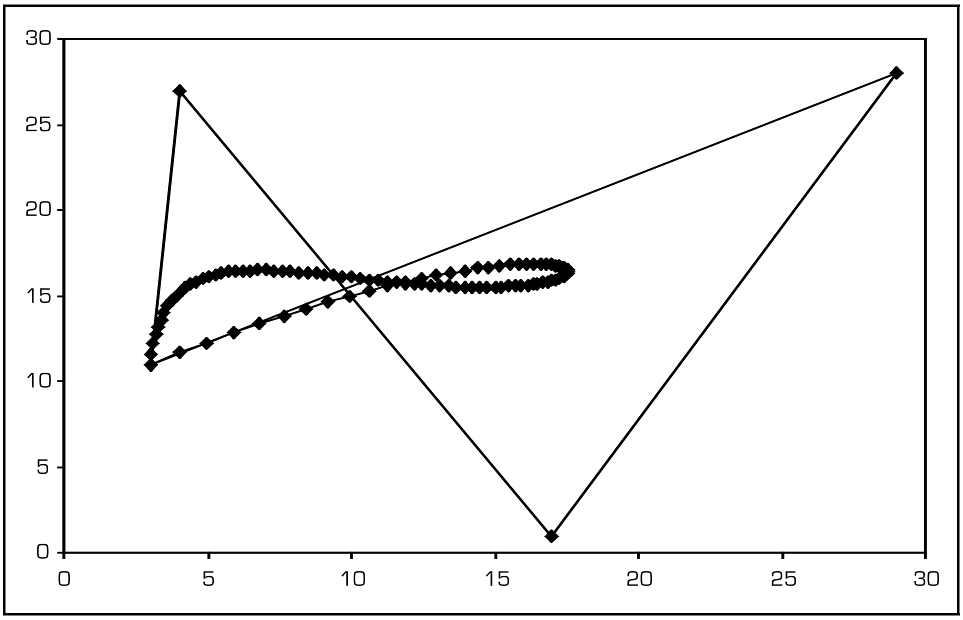

Figure 6. A Bezier curve with both terminals at (3, 11) and three control points.

Investigating Bezier curves adds a new dimension to the problem of curve sketching. You can play around with 3,4 and 5 point Beziers, or the Bernstein polynomials by going to the post-primary resources section on the Canberra Mathematics Association website www.canberramaths.org.au. There is also a host of websites available for deeper investigation, and perhaps my favourite is http://www.ablestable.com/play/bezier.

References

Devroye, L. (2005). *Bezier Curves.* Accessed 27 January 2005, at http://cgm.cs.mcgill.ca/~luc/bezier.html.

Gardner, M. (1959). *Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon and Schuster.